PROCESSES OF FORMATION OF HEXAGONAL CONVECTIVE CELLS

PMM Vol. 32, No. 2, 1968, pp. 244-255

Iu. B. PONOMARENKO
(Moscow)

(Received January 22, 1967)

In the present paper we study the development of convective instability in a horizontal fluid layer, leading to formation of hexagonal cells when the parameters are subject to some constraints. In addition we investigate the stability of the steady convective flows.

We know [1] that hexagonal convective cells may appear when a horizontal fluid layer is heated from below. It was shown however [2 to 9] that the hexagonal shape of the cells is not unique. In the recent years [5 to 8] it was found that the stability of convection with hexagonal cells depends on the effects, which are not taken into account when the convection is described in the usual manner [2 to 4 and 9 to 11], i.e. when the transport coefficients are temperature independent while the compressibility term appears in the equation of motion as a coefficient of volume expansion. In spite of the progress made it is still unclear, in what manner the processes of interaction of a large number of cumultative perturbations lead to formation of a periodic hexagonal structure and it is these processe, that we shall consider below.

1. When a fluid layer is heated from below, an equilibrium state can be reached in which the fluid is at rest, while the temperature, density and other hydrodynamic variables depend only on the vertical z-coordinate. This state may be unstable. In this case the deviation X of the hydrodynamic variables from their equilibrium values is given by [12]

$$X = \sum_k Ye^{ikr} \qquad Y = QZ + \sum Z_2(k_1, k_2)\, Q(k_1)\, Q(k_2) + \ldots \tag{1.1}$$

$$dQ/dt = \gamma Q + \sum H_2(k_1, k_2)\, Q(k_1)\, Q(k_2) +$$
$$+ \sum H_3(k_1, k_2, k_3)\, Q(k_1)\, Q(k_2)\, Q(k_3) + \ldots, \qquad Q(t=0) = \varepsilon A(k) \tag{1.2}$$

where k and r denote two-dimensional horizontal vectors. The wave vectors $k_1, \ldots, k_n$ appearing in the sums of order n in amplitude Q, satisfy the condition $k_1 + \ldots + k_n = k$.

We assume that the thickness of the layer is small compared with its horizontal dimension l, therefore the boundedness of the layer in the horizontal direction is only reflected in the fact that the components of the wave vector k assumed discrete values $2\pi n/l$ ($n = 0, \pm 1, \ldots$). All functions of the wave numbers are symmetrical in k and become the corresponding complex conjugates on changing the signs of k. Coefficients γ and H depend on the parameters λ, while the coefficients Z depend, in addition, on the vertical coordinate. The amplitude ε of the initial perturbation is assumed to be sufficiently small.

A simple example of computing H and Z is given in the Appendix. More complicated examples are given in [5 to 9], although in these cases some restrictions are imposed on the v lues of the wave vectors. In all these examples the magnitudes γ and H are real, thereiore in the following we shall assume that the properties of the systems in question are all real. We shall later show that for such a system Eq. (1.2) has a large number of the steady state, time-independent solutions, and that each such solution will describe some convective motion.

The rotational symmetry of the problem implies that the increment of the cumulative perturbations is $\gamma = \gamma(|k|)$. At the critical value of $\lambda = \lambda_*$, $\gamma = 0$ only for these perturbations, for which $|k| = k_*$. The wave vectors of such perturbations differ from each other only in the value of the polar angle φ

$$\varphi = {}^2/_3 \pi n / N \qquad (n = 0, 1, \ldots, 6N,\ 6N \approx k_* l)$$

In the following we shall assume that the number of perturbations is $6N \gg 1$.

Let us now suppose that the supercriticality is so small, that the increments γ of the perturbations which have $|k| = k_*$ are much smaller than the decrements of those perturbations for which $|k| \neq k_*$

$$|\gamma| \ll (2\pi / l)^2 |\gamma''(k_*, l_*)| \tag{1.3}$$

Then (1.2) can be reduced to an equation expressing the amplitude of the cumulative perturbations (whose wave vectors differ from each other only in the values of the polar angle φ), while the amplitudes of those perturbations for which $|k| \neq k_*$ can be expressed as functions of the amplitude of the cumulative perturbations

$$Q(k_1 + k_2) = \frac{2H_2(k_1, k_2) Q(k_1) Q(k_2)}{2\gamma - \gamma(k_1 + k_2)} \qquad (0 < |k_1 + k_2| \neq k_*) \tag{1.4}$$

$$Q(0) = \frac{2}{2\gamma - \gamma(0)} \sum_k H_2(k, -k)\, q(k)$$

$$\frac{dQ}{dt} = \gamma Q + a \bar{Q}_+ \bar{Q}_- - Q \sum_{\varphi'} b(\varphi, \varphi')\, q(\varphi') \tag{1.5}$$

$$Q(\varphi + \pi) = \bar{Q}(\varphi), \qquad q = Q\bar{Q}, \qquad a = 2H_2(k_+, k_-)$$

$$b = -6H_3(k', -k', k) - \frac{4H_2(k, 0) H_2(k', -k')}{2\gamma - \gamma(0)} - \frac{4H_2(k', k - k') H_2(k, -k')}{2\gamma - \gamma(k - k')}$$

Here and in the following we use the abbreviation $f_\pm(\varphi) = f(\varphi \pm {}^2/_3 \pi)$, and the summation is performed over all $0 \leqslant \varphi \leqslant 2\pi$. Moduli of the wave vectors appearing in the right-hand sides of (1.4) and (1.5) are all equal to k_*, and the last term in the expression for b should be omitted when $\varphi' = \varphi,\ \varphi \pm {}^1/_3 \pi$.

Since no direction is preferred in the system, we find that a is independent of φ, while $b = b(\varphi - \varphi')$. Moreover, the coefficient b satisfies

$$b(\psi) = b(-\psi) = b(\pi \pm \psi) \tag{1.6}$$

Indeed, each term

$$b(\varphi - \varphi')\, Q(\varphi)\, q(\varphi') \tag{1.7}$$

describes the interaction of three perturbations whose wave vectors have equal moduli and whose sum is equal to the given vector. Such three vectors are completely characterized by the acute angle ψ between the given vector and the straight line containing the remaining two vectors.

This angle has the same value for $\varphi - \varphi' = \pm\psi = \pi \pm \psi$, therefore the corresponding terms of (1.7) should coincide when $Q(\varphi) = \text{const}$.

The convection with hexagonal cells is described by the steady state solutions of (1.5) in which six symmetrically distributed perturbations have amplitudes of equal moduli, while the amplitudes of the remaining perturbations are equal to zero. When

$$a > 0, \qquad B_0 = \sum_{i=1}^{6} b({}^1/_3 \pi i) > 0$$

then one these solutions is

$$Q\left({}^1/_3\pi i\right) = {}^1/_2\left(a + \sqrt{a^2 + 4\gamma B_0}\right) / B_0 \qquad (i = 1, \ldots, 6) \tag{1.8}$$

In order to have the steady state amplitudes small, we must assume γ and a small (although their ratio may be arbitrary).

The increment γ is small near the boundary of stability. Under the laboratory conditions the value of a is also small [5 to 8], since $H_2 \neq 0$ only when the small magnitude effects (compressibility and the dependence of the transport coefficients on temperature) which are usually neglected [2 to 4 and 9 to 11], are taken into account. Since H_2 is small, we can assume in (1.5) that $b = -6H_3$.

The process of formation of the cellular convection has two distinct stages. During the first stage the cubic term in (1.5) is small, since the initial amplitude ε is small. Quadratic term describes the interaction of six, symmetrically distributed perturbations. The amplitudes of each group of six become infinite after a finite period of time, the period being determined by the form and amplitude of the initial perturbation (1.2) (in which $|k| = k*$). This interaction results in the appearance of six peaks on the functions $q(\varphi)$, and their distribution is determined by the form $A(\varphi)$.

In the second stage the cubic term is important. When $b > 0$, it restricts the growth of the perturbations, retaining in each peal only one perturbation with the maximum amplitude.

2 . Using the variables

$$Q = Re^{\gamma t}, \quad T = \gamma^{-1}\left(e^{\gamma t} - 1\right) \tag{2.1}$$

we can write the problem (1.5) and (1.2) as

$$\frac{dR}{dt} = \alpha \bar{R}_+ \bar{R}_- - (1 + \gamma T) R \sum_{\varphi'} b(\varphi - \varphi') r(\varphi') \qquad (r = R\bar{R},\ R_0 = \varepsilon A(\varphi)) \tag{2.2}$$

When $\epsilon \to 0$, the cubic term in (2.2) can be neglected for sufficiently small T, and the resulting problem has an exact solution. For the magnitudes

$$\bar{D} = a\left(RR_+R_-\right), \quad E = {}^1/_2\left(D - \bar{D}\right), \quad F = {}^1/_2\left(D + \bar{D}\right), \quad S = {}^1/_3\left(r + r_+ + r_-\right),$$
$$P = {}^1/_3\left(rr_+ + r_+r_- + r_-r\right) \tag{2.3}$$

we can easily obtain, from (1.5), the following relation (a prime denotes a derivative in T

$$\bar{R}R' = D, \qquad D' = 3a^2 P, \quad E = {}^1/_2 r d \ln\left(R / \bar{R}\right) / dT = E_0 \tag{2.4}$$
$$r - r_0 = S - S_0, \quad S' = 2F, \quad F' = 3a^2 P, \quad P' = 4SF$$

The subscript zero means that the relevant magnitude is taken at $T = 0$. Eqs. (2.4) have the following integrals

$$\theta = \arg R = {}^1/_2 \ln \frac{R}{\bar{R}} = \theta_0 + E_0 \int_0^T \frac{d\tau}{r(\tau)}$$
$$S^2 - P = \left(S^2 - P\right)_0 = S_*^{\,2}$$

Thus the system (2.4) can be reduced to

$$S' = 2F, \qquad F' = 3a^2\left(S^2 - S_*^{\,2}\right) \tag{2.5}$$

Fig. 1 shows the phase plane of the system (2.5). Arrows denote the direction of motion and we see that the phase curves are symmetrical with respect to the S-axis.

The integral of (2.5) can conveniently be written as

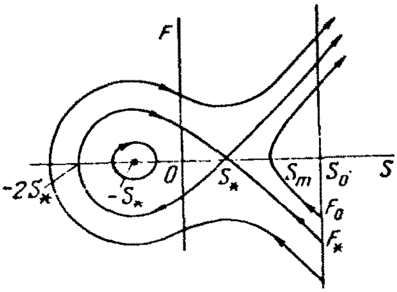

Fig. 1

$$F^2(S, S_m) = a^2(S - S_m)(S^2 + SS_m + S_m^2 - 3S_*^2)$$
$$F_0 = F(S_0, S_m), \qquad F_* = F(S_0, S_*) \tag{2.6}$$

Obviously, the magnitudes S and r are non-negative for any T. This means that (Fig. 1) we have

$$F_*^2 - F_0^2 \geqslant 0, \qquad \min(r, r_+, r_-)_0 + S_m - S_0 \geqslant 0$$

for any initial conditions.

We shall give one of the direct proofs of the above inequalities in the Appendix.

Solution of (2.5) and (2.6) is given by

$$J(S) = \int_{S_m}^{S} \frac{dS}{\sqrt{(S - S_m)(S^2 + SS_m + S_m^2 - 3S_*^2)}} \tag{2.7}$$

When $S \geq S_0$, it has the form

$$J(S) \mp J(S_0) = 2|a|T \tag{2.8}$$

where the upper sign is used when $F_0 > 0$, and the lower sign— when $F_0 < 0$.

The value of S becomes infinite after the time $T_\infty = T(S = \infty) \sim 1 / |a\varepsilon|$. When $T_\infty - T$ are small, (2.7) and (2.8) yield

$$2|a|(T_\infty - T) = J(\infty) - J(S) = \frac{2}{\sqrt{S}}\left[1 + O\left(\frac{\varepsilon^4}{S^2}\right)\right] \tag{2.9}$$
$$r = S = a^{-2}(T_\infty - T)^{-2}, \qquad F = a^{-3}(T_\infty - T)^{-3}$$

Behavior of the amplitude q depends essentially on the sign of $V = 1 + \gamma T_\infty$

$$q = \begin{cases} (1 + \gamma T)^2 r(-1/\gamma) & (T \to -1/\gamma, V < 0) \\ (V/a)^2/(T_\infty - T)^2 & (T \to T_\infty, V > 0) \\ \gamma^2/a^2 & (T \to T_\infty, V = 0) \end{cases}$$

The magnitudes F, S, P, T_∞ introduced above are $1/3\pi$-periodic functions of φ. Let us now assume that the form of the initial deviation $A(\varphi)$ is such, that the time $T_\infty(\varphi)$ on the interval $0 \leqslant \varphi < 1/3\,\pi$, attains its minimum value T_* at an unique point φ_*; then, for small $T_* - T$, the function $r(\varphi)$ will have six sharp maxima at the points $\varphi = \varphi_* + 1/3\,\pi i\ (i = 0, \ldots, 5)$. The amplitude $q(\varphi)$ will also exhibit such maxima, provided that $V_* = 1 + \gamma T_* > 0$ (when $V_* < 0$, all perturbations cease at large t).

3. Under certain conditions (defined in this section) a characteristic time T_0 exists, at which the function $q(\varphi)$ has already formed sharp peaks, while the cubic term in (2.2) is still small.

The width Δ of the peaks is given by

$$6r_*\Delta = \frac{\pi}{3N}\Sigma r \approx \int_0^{2\pi} r(\varphi)\,d\varphi, \qquad r_* = r(\varphi_*) \tag{3.1}$$

When $T \to T_*$, (3.1) and (2.9) yield

$$\Delta = {}^1/_6 \int_0^{2\pi} d\varphi \left(\frac{T_* - T}{T_\infty - T}\right)^2 \approx \int_{-\infty}^{\infty} d\varphi \left(1 + \frac{{}^1/_2\varphi^2 T_*''}{T_* - T}\right)^{-2} = \pi\left(\frac{T_* - T}{2T_*''}\right)^{1/2} \tag{3.2}$$

The peaks appear sharp when $\Delta \ll 2\pi$. We shall use

$$\sum_{\varphi'} b(\varphi - \varphi')\, r(\varphi') = B(\varphi)\, r_*\Delta = {}^1/_6 B(\varphi)\,\Sigma r, \qquad B(\varphi) = \sum_{i=1}^{6} b({}^1/_3\pi i + \varphi - \varphi_*) \tag{3.3}$$

to estimate the cubic term in (2.2). The accuracy of this equation is inversely proportional to Δ. The cubic term restricts the growth of perturbations, provided that $B(\varphi) > 0$. Taking this term into account we can obtain from (2.2) and (3.3)

$$r' = 2F - 2(1 + \gamma T)\, r r_* B(\varphi)\Delta \tag{3.4}$$

which, together with (2.9), implies that when $\varphi = \varphi_*$, then the cubic terms is small compared with its derivative, provided that

$$\Delta(T) \gg V_* B_* / (a^2 T_*''), \qquad B_* = B(\varphi_*) \tag{3.5}$$

The characteristic time T_0 mentioned above exists, if

$$V_* B_* / (a^2 T_*'') \ll \Delta(T_0) \ll 1 \tag{3.6}$$

When $\gamma > 0$ and $T_*'' \sim T_* \sim 1 / |a\varepsilon|$ the inequalities (3.6) hold, provided that each of the magnitudes

$$B_*\varepsilon / |a|, \qquad \gamma B_* / a^2$$

is small compared with unity; if $\gamma < 0$, then the assumption that the sum of these magnitude is small, suffices.

4. Let the inequalities (3.6) hold. Then we find, that when $t \geq t_0 = \ln(1 + \gamma T_0)/\gamma$, Eq. (1.5) becomes (with (3.3) taken into account)

$$dQ / dt = \Gamma Q + a\overline{Q}_+\overline{Q}_-, \qquad \Gamma = \gamma - {}^1/_6 B(\varphi)\,\Sigma q \tag{4.1}$$

Its solution is expressed in the terms of the following functions:

$$U = \exp\int_{t_0}^{t} \Gamma(t, \varphi)\, dt, \qquad \tau = T_0 + \int_{t_0}^{t} U dt \tag{4.2}$$

which themselves constitute a solution of the following problem

$$dU / dt = U\Gamma, \quad d\tau / dt = U, \quad U(t_0) = 1, \quad \tau(t_0) = T_0 \tag{4.3}$$

From (4.1) and (4.2) it follows that $R = Q/U$ satisfies Eq. $dR/d = a\overline{R}_+\overline{R}_-$, whose solution was discussed in Section 3. According to (2.9) we have, for $\varphi \approx \varphi_*$

$$FU^3 = y^3 / a^2, \qquad q = U^2 r = y^2 / a^2, \qquad y = U / (T_\infty - \tau) \tag{4.4}$$

From (4.3) we have

$$dy / dt = y\Gamma + y^2, \ y(t_0) = 1 / (T_\infty - T_0), \ \Gamma = \gamma - {}^1\!/_6 B(\varphi) W_2 / a^2, \ W_m = \Sigma y^m \tag{4.5}$$

The amplitude Q can be found if $W_2(t)$ is known. To obtain it, we shall multiply (4.5) by my^{m-1} and sum the result over all φ

$$dW_m / dt = m(\Gamma_* W_m + W_{m+1}) \qquad (m = 2, \ldots, M-1) \tag{4.6}$$

Here we have utilised the relation

$$\Sigma \Gamma y^m = \Gamma_* W_m \tag{4.7}$$

which holds for at least small values of $t - t_0$ when the peaks on the function y are sufficiently sharp. At any M we find that the number of equations in (4.6) is one less than the number of the unknowns. Thus the system (4.6) together with (4.5) becomes a closed one for $\varphi = \varphi_*$ provided that we set the value of M sufficiently large and put $W_M = y_*{}^M$.

Eqs. (4.6) and their initial conditions are, together, equivalent to the problem (4.3) in which B is independent of φ and equal to B_*. Solution y_*, $T_* = T$ of this problem can be found directly.

Relations (4.3) and (4.4) yield

$$\frac{dU_*}{dT} = \Gamma_* = \gamma - \frac{B_* U_*^2}{6a^2} \sum (T_\infty - T)^{-2} \tag{4.8}$$

For $\gamma = 0$ the solution becomes

$$\frac{1}{U_*} - 1 = \frac{B_*}{6a^2} \sum \left(\frac{1}{T_\infty - T} - \frac{1}{T_\infty - T_0} \right) \qquad (T_0 \leqslant T \leqslant T_*)$$

from which it follows that when $T \to T_*$,

$$U_* = u(T_* - T)\{1 + O[(T_* - T)^2 / (T_1 - T_*)^2]\}, \ u = a^2 / B_*,$$

$$T_1 = \min T_\infty (\varphi_* \pm {}^1\!/_3 \pi / N) \tag{4.9}$$

When $\gamma \neq 0$, the asymptotic solution has, as before, the form (4.9), but u is given by

$$u = {}^1\!/_2 (1 + \sqrt{1 + 4\gamma B_* / a^2})\, a^2 / B_*$$

Relations (4.3), (4.4) and (4.9) readily yield

$$|Q_*| \to u / |a|, \qquad F_* U_*^3 \to u^3 / a^2 \quad \text{as} \quad T \to T_* \tag{4.10}$$

From (2.3) it follows that

$$F_* U_*^3 \to a \cos(\theta + \theta_+ + \theta_-)_* (u / |a|)^3$$

which, together with (4.10), implies that

$$\cos(\theta + \theta_+ + \theta_-)_* \to a / |a| \tag{4.11}$$

Thus, in the steady state, the phases of the amplitudes Q become partly correlated, while their exact value can be found from the initial conditions and are given by the expression for θ (see above).

The angle φ_* together with three phases θ in (4.11) define the orientation of the hexagons relative to the initial coordinate system (on which the initial deviation was given).

This system can be rotated and translated in the horizontal plane in such a manner, that the coordinate origin will coincide with the center of one of the hexagons, and one of the horizontal axes will pass through two vertices of this hexagon. Rotation will be defined by the angle φ_*, while the translation will be given by three phases satisfying the relation (4.11). As expected, the orientation of the hexagons is determined by the form of the initial deviation.

Evolution of the perturbations with $\varphi \neq \varphi_*$ can be described by the relations (4.4) and (4.2) in which $W_2\,(t)$ is obtained from (4.3) and (4.8)

$$\frac{W_2}{6a^2} = \frac{\gamma - dU_*/dT}{B_*}, \qquad \int\limits_{T_0}^{T} \frac{dT}{U_*(T)} = t - t_0 \tag{4.12}$$

Relations (4.2) and (4.12) yield

$$\frac{U(\varphi, t)}{U_*} = \exp\left(\frac{B_* - B}{6a^2}\int\limits_{t_0}^{t} W_2\,dt\right) = \left(\frac{\exp\gamma(t - t_0)}{U_*}\right)^{1-B/B_*}$$

By (4.4), all perturbations with $\varphi \neq \varphi_*$

$$\tau(\varphi, \infty) = T_0 + \int\limits_{t_0}^{\infty} U(\varphi, t)\,dt < T_\infty(\varphi) \tag{4.13}$$

As $t \to \infty$, we have

$$T_* - T \to e^{-ut}, \qquad U \to u^{B/B_*}\exp\left(-ut\,[1 - (B_* - B)u/a^2]\right)$$

Therefore the sufficient condition for (4.13) to hold is, that the expression within the square brackets is positive for any φ. At the same time (4.13) will hold, if the form of the initial deviation is such, that the minimum T_∞ at the point φ_* is sufficiently sharp. If the condition (4.13) is violated, then at sufficiently large t the relations (4.7) and (4.8) cease to hold since additional sextuple peaks whose amplitudes are comparable with the amplitudes of the fundamental set of peaks, begin to appear on the function $q\,(\varphi)$. The resulting steady state of the system will have several sextuple amplitudes different from zero, and in accordance with (4.4) we may expect that each sextuple set will have equal amplitudes and that $F > 0$.

5. When considering the formation processes, we have imposed strong constraints (3.6) on the parameters of the system. The steady state may also be stable under the weaker constraints. In order to determine them, we shall investigate the stability of the symmetric, steady solutions of (1.5). In such solutions, the only amplitudes different from zero, are

$$Q(\varphi_0 + i\pi/n) \qquad (i = 1, \ldots, 2n,\ n \geqslant 1) \tag{5.1}$$

Solutions (5.1) in which only the angle φ_0 and the phases of the amplitudes Q are different, are physically indistinguishable, therefore in the following we shall assume that $\varphi_0 = 0$ and, that the amplitudes Q are real. It can be assumed without loss of generality that in (1.5) $a \geq 0$ and, that the steady-state amplitudes are positive (if $a < 0$, then in accordance with (4.11) we can assume that when $n/3$ the amplitudes (5.1) is an integer, for which $i/3$ are integers, are negative). Henceforth we shall only consider those steady states, which can be stable.

The following notation is used in investigating the stability of (5.1):

$$B_n(\varphi) = \sum_{i=1}^{2n} b(\varphi + i\pi/n), \quad \beta_n = B_n(0), \quad \alpha_n = B_n(^2/_3\pi), \quad h_n = \frac{1}{1+\sqrt{1+4\gamma\beta_{3n}/a^2}}$$

From (1.6) it follows that

$$B_n(\varphi) = B_n(-\varphi) = B_n(\pi/n + \varphi) = 2n \sum_{m=0}^{\infty} b_{mn} \cos 2mn\varphi \tag{5.2}$$

which, after putting $n = 1$, yields the relations for $b = \frac{1}{2}B_1$. Conditions of stability depend, essentially, on the divisibility of n by 3. Let us suppose that n is not divisible by 3; then in the steady-state the quadratic term in (1.5) vanishes and we have the following expression for the steady state amplitudes Q_0.

$$Q_0 = \sqrt{\gamma/\beta_n} \qquad (\gamma \geqslant 0, \quad \beta_n > 0) \tag{5.3}$$

Infinitesimal perturbations $Q(\varphi)$ of the steady state solution (5.1) and (5.3) satisfy

$$Q^{\cdot} = Qq_0(\beta_n - B_n), \qquad \varphi \neq i\pi/n, \quad i\pi/n + {}^2/_3\pi \tag{5.4}$$

which is obtained by linearising (1.5) in Q, and which yields one of the conditions of stability

$$\beta_n < B_n(\varphi) \qquad (\varphi \neq i\pi/n, \quad i\pi/n + {}^2/_3\pi) \tag{5.5}$$

Perturbations $Q_{\pm} = Q(i\pi/n \pm {}^2/_3\pi)$ satisfy Eq.

$$Q_{\pm}^{\cdot} = \gamma Q_{\pm} + \alpha Q_0 Q_{\mp} - \alpha_n Q_{\pm}$$

Adding and substracting these equations we obtain the additional conditions of stability

$$\gamma - q_0\alpha_n + aQ_0 < 0, \qquad \gamma/a^2 > \beta_n/(\alpha_n - \beta_n)^2 \tag{5.6}$$

Equations for the pertubations $Q_i = Q(i\pi/n)$ have the form

$$q_i^{\cdot} = -2q_0 \sum_{i=1}^{n} b_{ij}q_i \quad (1 \leqslant i \leqslant n), \qquad b_{ij} = b(\pi(i-j)/n) = b_{|i-j|,n} = b_{n-|i-j|,n} \tag{5.7}$$

from which it follows that the steady-state solution is table, if all roots λ of Eq.

$$D_n \equiv |\lambda\delta_{ij} + b_{ij}| = 0 \tag{5.8}$$

are negative.

Each row of the determinant (5.8) is the same cyclic permutation of the one above (such determinant is of the circulant type).

The circulant type determinant D_n whose first row elements are $a_1, \ldots, a_n$, is equal to [13]

$$f(\varepsilon)f(\varepsilon^2)\ldots f(\varepsilon^n) \qquad (f(x) = a_1 + a_2x + \ldots + a_nx^{n-1}, \qquad \varepsilon = e^{2\pi i/n})$$

For the determinant (5.8) we have

$$f(\varepsilon^m) = \lambda + \sum_{s=1}^{n} b\left(\frac{s\pi}{n}\right)\cos\frac{2\pi ms}{n} \equiv \lambda + \rho(m, n) \tag{5.9}$$

Since $\rho(m, n) = \rho(n - m, n)$, the stability conditions are

$$\rho(m, n) > 0 \quad (0 \leqslant m \leqslant {}^1/_2\, n) \tag{5.10}$$

When n is large, the Fourier expansion (5.2) may be limited to its first two terms, while the sum in (5.9) can be replaced by an integral; the conditions (5.5), (5.6) and (5.10) will then become

$$b_n < 0, \quad \gamma / a^2 > 2 b_0 / (n b_n^2); \quad b_m > 0, \quad 0 \leqslant m \leqslant {}^1/_2 n \qquad (5.11)$$

Since the first and third condition have an opposing sense, it follows that the number of stable solutions is finite for any functions $b(\varphi)$. In particular, if $b(\varphi) > b(0)$, then the conditions (5.5) and (5.10) hold [9] only for the solution with $n = 1$.

The second condition of (5.11) shows that the region of stability decreases with increasing n (since the Fourier coefficients b_n decrease faster than $1/n$).

For the steady-state solutions (5.1) in which the number is equal to $3n$, the quadratic term in (1.5) is different from zero and

$$Q_0 = \frac{1}{2} a / (h_n \beta_{3n}) \qquad (h_n < 1, \qquad \beta_{3n} > 0) \qquad (5.12)$$

Perturbation Eq. has the form

$$Q^{\cdot} = Q(\gamma - Q_0^2 B_{3n})$$

This yields one of the stability conditions

$$2 h_n > 1 - B_3 n / \beta_{3n} \qquad (5.13)$$

Perturbations $Q_i = Q({}^1/_3 \, i\pi / n)$ satisfy

$$Q_i^{\cdot} = a Q_0 (-Q + \overline{Q}_+ + \overline{Q}_-)_i - Q_0^2 \sum_{j=1}^{6n} X_i b_{ij} \qquad (1 \leqslant i \leqslant 6n) \qquad (5.14)$$

$$X_i = (Q + \overline{Q})_i, \qquad b_{ij} = b\left(\frac{1}{3} \pi (i - j) / n\right)$$

from which we have

$$X_i^{\cdot} = a Q_0 (S_i - 2 X_i) - 4 Q_0^2 \sum_{j=1}^{3n} X_j b_{ij}, \qquad S_i = (X + X_+ + X_-)_i \qquad (5.15)$$

From this we obtain, for S

$$S_i^{\cdot} = a Q_0 S_i - 2 Q_0^2 \sum_{j=1}^{n} B_{ij} S_j \qquad (i = 1, \ldots, n) \qquad B_{ij} = B_3({}^1/_3 \pi (i - j) / n)$$

Solution of this equation will decay, if the roots λ of

$$|(\lambda - {}^1/_2 a / Q_0) \delta_{ij} + B_{ij}| = 0 \qquad (5.16)$$

are negative.

The determinant (5.16) is a circulant, consequently the additional stability conditions are obtained in the form

$$h_n < \frac{\sigma(m, n)}{\beta_{3n}} \quad \left(1 \leqslant m \leqslant \frac{1}{2} n\right), \quad \sigma(m, n) = \sum_{s=1}^{n} B_3\left(\frac{s\pi}{3n}\right) \cos \frac{2\pi m s}{n} \qquad (5.17)$$

If they are satisfied, then in (5.15) $S \to 0$ at large t. The magnitudes X decay, if the roots λ of

$$|(\lambda + {}^1/_2 a / Q_0) \delta_{ij} + b_{ij}| = 0 \qquad (i, j = 1, \ldots, 3n) \qquad (5.18)$$

are negative.

The determinant (5.18) is a circulant and the additional stability conditions have the form

$$h_n > -\rho(m, 3n)/\beta_{3n} \qquad (1 \leqslant m \leqslant {}^3/_2 n) \tag{5.19}$$

Equations for $Y_i = Q_i - \overline{Q}_i$ do not lead to additional stability conditions. Their solution

$$Y_i(t) = Y_i(0) - {}^1/_3\,[1 - \exp(-3aQ_0 t)][Y(0) + Y_+(0) + Y_-(0)]_i$$

shows that as $t \to \infty$, the phase perturbations do not vanish although (4.11) hold for each set of six.

At large n the conditions (5.12), (5.13), (5.17) and (5.19) can be expressed in terms of the coefficients appearing in (5.2)

$$\beta_{3n} \doteq 6nb_0 > 0 \quad (0 < h_n < 1), \qquad 2b_0 h_n > b_{3n} + |b_{3n}|$$

$$2b_0 h_n < b_{3m} \quad (1 \leqslant m \leqslant {}^1/_2 n), \qquad 4b_0 h_n > -b_m \quad (1 \leqslant m \leqslant {}^3/_2 n)$$

and it follows from these conditions, that the necessary condition for stability is, that $b_{3m} > 0$.

Hexagonal convective cells are stable, if (5.20)

$$\beta_3 = 2b(0) + 4b({}^1/_3\,\pi) > 0, \quad \beta_3 h_3 > b({}^1/_3\,\pi) - b(0), \quad 1 > h_3 > {}^1/_2\,(1 - \min B_3/\beta_3)$$

(all these conditions except the last one, were obtained earlier in [6]).

In the actual problems [9] (see also the Appendix) $b(\varphi) > 0$, therefore the region of stability exists on the axis γ/a^2.

6. The steady state solutions of (1.5) are, in general, nonsymmetric. They consist of m sets of six and n (not included in the six-sets) pairs of positive amplitudes, and within each six-set the amplitudes are equal to each other, while the angles φ_i, for which $Q_i = Q(\varphi_i) > 0$, are not proportional to i. The condition $Q_i > 0$ imposes a restriction on the possible values of φ_i.

Conditions of stability of the nonsymmetric solutions are obtained in the same manner as the corresponding conditions for the symmetric solutions. For example, instead of (5.5) we can find

$$B(\varphi) = \sum_{j=1}^{6m+2n} b(\varphi - \varphi_j)\, q_j > B(\varphi_*)$$

where φ_i characterized the steady state amplitudes appearing in n pairs. Similarly we can find the analogs of (5.6) and (5.13) by considering the perturbations $Q(\varphi \neq \varphi_i)$. We find that the matrix $|B(\varphi_i - \varphi_j)|$ in the equations describing the perturbations $Q(\varphi_i)$ is symmetric, but not circulant; therefore the corresponding stability conditions cannot be obtained explicitly for every n and m.

When discussing the formation processes we noted, that the nonsymmetric solutions may be obtained when the initial perturbation is of the suitable form (when the function $T_\infty(\varphi)$ introduced above has a weak minimum). It may also occur, when min T_∞ occurs at two distinct points

Finally we note, that in deriving (1.5) we have utilised the fact that the fluid layer is bounded in the horizontal directions, although the restriction (1.3) imposed on the supercriticality is not essential $\varphi = \varphi_* \pm {}^1/_6\,\pi / N$).

In general we shall assume that the wave vectors appearing in the right-hand sides of (1.4) and (1.5) have their moduli equal to that value of $k_0(\lambda)$, for which γ is maximum. It may happen that the denominator in (1.4) may become zero at some particular values of λ. In this case we may expect [12] that a steady-state motion will be set up in the system, with the wave vectors whose moduli will differ from each other by about $1/l$.

In the limit, as $l \to \infty$, solution of the system (1.1) and (1.2) may be sought in the form of a series in ε with subsequent summation of the asymptotic (at large t) terms of the series [12].

Appendix. As an example we shall obtain (1.1) and (1.2) for a simple model problem [6].

$$\left[\frac{\partial}{\partial t} - \left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right)^3 + \lambda \frac{\partial^2}{\partial r^2} - 2\nu \cos z\right] X + \frac{1}{2} \frac{\partial^3 X^2}{\partial z^3} = 0$$

$$X = \partial^2 X / \partial z^2 = \partial^4 X / \partial z^4 = 0 \quad \text{when} \quad z = 0, \pi \tag{A.1}$$

where ν is a small parameter. Inserting (1.1) into (A1) we obtain

$$Y^{\cdot} - LY + {}^1/_2 \Sigma (Y_1 Y_2)''' = 0, \qquad Y_i = Y(k_i) \tag{A.2}$$

$$L = \lambda k^2 + (\partial^2 / \partial z^2 - k^2)^3 + 2\nu \cos z \qquad (k_1 + k_2 = k)$$

Here the dot denotes differentiation with respect to t, while the dash – differentiation with respect to z.

Boundary conditions for Y are the same as those for X.

Equations for γ, H_n and Z_n are obtained by inserting (1.1) and (1.2) into (A2), writing the result as a series in Q of the type of (1.1), and equating to zero the coefficients of like Q in the sums. Boundary conditions for Z_n are the same as those for X.

Equation for γ and Z has the form

$$\gamma Z - LZ = 0$$

For the unstable branch we have

$$Z = \sin z + \nu K \sin 2z + O(\nu^2) \tag{A.3}$$

$$\gamma = \lambda k^2 - (1 + k^2)^3 + O(\nu^2), \qquad K^{-1} = (4 + k^2)^3 - (1 + k^2)^3$$

At some $k_0(\lambda)$, the increment will assume its maximum value γ_0. With small supercriticality $\Lambda = \lambda - {}^{27}/_4$ we have

$$|k_0| = (1 + {}^1/_9\Lambda) / \sqrt{2}, \qquad \gamma = {}^1/_2\Lambda - 9(|k| - k_0)^2 \tag{A.4}$$

Equation for Z_2 has the form

$$(\gamma_1 + \gamma_2 - L) Z_2 = - Z H_2 + 2 \sin 2z - {}^1/_4 \nu (K_1 + K_2)(\sin z - 27 \sin 3z) + O(\nu^2) \equiv \Psi$$

$$(k_1 + k_2 = k, \quad \gamma_i = \gamma(k_i), \quad K_i = K(k_i))$$

A solution exists for any k, if

$$\int_0^\pi dz Z \Psi = 0$$

This, together with (A3), yields

$$H_2 = - {}^1/_4 \nu (K_1 + K_2 - 8K) + O(\nu^2), \qquad k = k_1 + k_2 \tag{A.5}$$

$$Z_2 = 2V \sin 2z + O(\nu), \qquad V^{-1} = \gamma_1 + \gamma_2 - \lambda k^2 + (4 + k^2)^3$$

When determining Z_3, we can assume that $\nu = 0$. The corresponding equation has the the form

$$(\gamma_1 + \gamma_2 + \gamma_3 - L) Z_3 = - {}^1/_6 [V(k_1, k_2) + V(k_2, k_3) + V(k_3, k_1) + 6H_3](\sin z - 27 \sin 3z) - 27 H_3 \sin 3z \qquad (k_1 + k_2 + k_3 = k)$$

To find H_3 we must equate to zero the expression within the square brackets. This, together with (1.4) and (A3) to (A5) will yield, with the accuracy up to the terms

$$K = 4/351, \qquad a = 3\nu K$$

$$b = \frac{1}{64} + \frac{1}{(5 + c)^3 - {}^{27}/_4 (1 + c)} + \frac{1}{(5 - c)^3 - {}^{27}/_4 (1 - c)}, \qquad c = \cos(\varphi - \varphi')$$

Function $b(c)$ is even and satisfies (1.6). Since $b > 0$ we find, in accordance with (5.20), that a region of stability of hexagonal cells exists.

In Section 3 we used physical considerations to derive less obvious inequalities. It is sufficient to prove them for $F_0^2 = (rr_+r_-)_0$, $r_0 = 1$. When proving the first inequality we find it convenient to assume, that $0 \leqslant r_\pm \leqslant 1$ (in the second inequality $r_\pm \geqslant 1$) is more convenient).

It appears that the symmetric functions of the arguments $r_\pm$ (which are shown to be nonnegative) can exhibit extrema only on the straight line $r_+ = r_-$. After that we confirm that the inequalities hold on this straight lines as well as on the boundary of the region of variation of $r_\pm$.

BIBLIOGRAPHY

1. Benard, M., Les tourbillous cellulaires dans une nape liquide Transportant de la chaleur par convection en régime permanent. Ann. Chim. Phys., Vol. 23, p. 62, 1901.
2. Chandrasekhar, S., Hydrodynamic and Hydromagnetic Stability. Oxford, Clarendon Press, 1961.
3. Gorkov, L.P., Steady convection in the two-dimensional fluid layer near the critical mode of heat transfer. ZhETF, Vol. 33, No. 2, 1957.
4. Malkus, W. and Veronis, G., Finite amplitude cellular convection. J. Fluid Mech., Vol. 4, pt. 3, 1958.
5. Palm, E., On the tendency towards hexagonal cells in steady convection. J. Fluid Mech., Vol. 8, pt. 2, 1960.
6. Segel, L.A. and Stuart, T.T., On the question of the preferred mode in cellular thermal convection. J. Fluid Mech., Vol. 13, pt. 2, 1962.
7. Segel, L.A., The structure nonlinear cellular solution to the Boussinesq equations. T. Fluid. Mech., Vol. 21, pt. 2, 1965.
8. Segel, L.A., The nonlinear interaction of a finite number of disturbances to a layer of fluid heated from below. J. Fluid Mech., Vol. 21, pt. 2, 1965.
9. Schlüter, A., Lortz, D. and Busse, F., On the stability of steady finite amplitude convection. J. Fluid. Mech., Vol. 23, pt. 1, 1965.
10. Boussinesq, J., Théorie Analytique de la Chaleur Paris, Gauthier – Villars, 1903.
11. Landau, L.D. and Lifshits, E.M., Mechanics of Continuous Media. 2nd. ed. M., Gostekhizdat, 1954.
12. Ponomarenko, Iu. B., Occurrence of space – periodic motions in hydrodynamics. PMM, Vol. 32, No. 1, 1968.
13. Mishina A.P. and Proskuriakov I.V., Higher Algebra. M., Fizmatgiz, 1962.

Translated by L.K.